

Hofstadter problem in higher dimensions

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We investigate some generalizations of the Hofstadter problem to higher dimensions with Abelian and non-Abelian gauge field configurations. We numerically show the hierarchical structure in the energy spectra with several lattice models. We also point out the equivalence between the π -flux state and the staggered formalism of the Dirac fermion.
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Subject Index B01, B38

1. Introduction

The fractal structure of the energy spectrum for the two-dimensional magnetic lattice system, known as Hofstadter's butterfly [1], is one of the most exotic consequences of the quantum property of low dimensionality. Such a fractal nature of the magnetic system can be observed in a simple tight-binding model in the presence of a magnetic field. This Hofstadter problem and its variants have been extensively discussed in various areas of physics and also mathematics. More recently it can be realized even in experimental situations [2,3].

In this paper we extend the Hofstadter problem, which is originally considered in two dimensions, to higher dimensions with not only Abelian, but also non-Abelian, gauge field configurations. So far there have been some attempts to generalize it to the three-dimensional magnetic system [4–7], and also to that in non-Abelian gauge potential [8–12]. But its generalization to a much higher-dimensional system has not yet been studied in the literature. Actually, when we analyze topological matters in the lower-dimensional system, the four-dimensional point of view can be quite useful: topological insulators/superconductors in two and three dimensions [13,14] are deeply connected to the four-dimensional quantum Hall effect (QHE) [15] through the dimensional reduction procedure [16,17]. The formalism discussed in this paper can be applied to arbitrary even-dimensional lattice systems in the presence of a magnetic field. Based on this formalism, we prove that the π -flux state [18,19], which has a gapless excitation in general, is essentially equivalent to the Dirac fermion in arbitrary dimensions. Furthermore, when we apply a non-Abelian gauge field, we can obtain a hierarchical structure in the energy spectrum of the lattice model even in higher dimensions.

This paper is organized as follows. In Sect. 2 we introduce arbitrary even-dimensional lattice models with Abelian gauge field background configuration. We discuss the corresponding Schrödinger equation to the magnetic system, and obtain Harper's equation. We then comment on its connection to the non-commutative torus. We also provide a general proof for the gapless spectrum of the π -flux state, by referring to its equivalence to the naively discretized lattice Dirac fermion. In Sect. 3 we then consider the Hofstadter problem for the non-Abelian gauge field configuration. In particular,

the four-dimensional SU(2) theory is investigated as a fundamental example of non-Abelian gauge theory. We show some numerical results of the model, and discuss the effect of the inhomogeneity of the background flux. Section 4 is devoted to a summary and discussion.

2. Abelian gauge field models

The first generalization of the Hofstadter problem is formulated in arbitrary even dimensions in the presence of the U(1) gauge field. Before introducing a lattice model, we now consider the following background U(1) field configuration for the continuum theory [20],

$$F_{2s-1,2s} = \omega_s, \quad F_{\mu\nu} = 0 \quad \text{for otherwise, } (s = 1, \dots, r). \quad (1)$$

We then apply the higher-dimensional version of the Landau gauge to this configuration,

$$A_{2s-1}(x) = -\omega_s x_{2s}, \quad A_{2s}(x) = 0. \quad (2)$$

Here the field strengths ω_s are quantized,

$$\omega_s = \frac{2\pi}{L_{2s-1}L_{2s}} n_s, \quad n_s \in \mathbb{Z}, \quad (3)$$

and thus the topological number is given by

$$Q = \frac{1}{2^r \cdot r!} \int d^{2r}x \epsilon^{\mu_1 \dots \mu_{2r}} \text{Tr} F_{\mu_1 \mu_2} \dots F_{\mu_{2r-1} \mu_{2r}} = \prod_{s=1}^r n_s. \quad (4)$$

This is regarded as the r th Chern number.

To discuss the Hofstadter spectrum, we then realize these configurations on the lattice. The gauge potential (2) is implemented by introducing the link variable, which can be regarded as the Wilson line,

$$U_{2s-1}(x) = e^{-i\omega_s x_{2s}}, \quad U_{2s}(x) = 1. \quad (5)$$

Precisely speaking, we have to assign appropriate boundary conditions even for $U_{2s}(x)$ [20]. In this case the field strength is restricted to the interval $(0, 2\pi)$ due to the lattice discretization [21,22]. Furthermore, they are characterized by the following fractions,

$$\frac{p_s}{q_s} \equiv \frac{n_s}{\tilde{L}_{2s-1} \tilde{L}_{2s}}. \quad (6)$$

Here, \tilde{L}_i is related to the system size as $L_i = \tilde{L}_i a$ where a is the lattice spacing. Note that they satisfy $0 < p_i/q_i < 1$. This fraction plays an essential role in the interesting spectrum of the model we discuss below.

2.1. Tight-binding model

The lattice Hamiltonian with this background configuration is defined as

$$\mathcal{H}_{\text{tight}} = \sum_{x,\mu} \left[c_{x+\hat{\mu}}^\dagger U_\mu(x) c_x + c_x^\dagger U_\mu^\dagger(x) c_{x+\hat{\mu}} \right]. \quad (7)$$

This is just the tight-binding hopping model, which describes the non-relativistic particle with the background *magnetic* field. Introducing the state $|\psi\rangle = \sum_x \psi_x c_x^\dagger |0\rangle$, we obtain the corresponding

Schrödinger equation from the Hamiltonian (7) written in a second quantized form,

$$\sum_{\mu=1}^{2r} \left[U_{\mu}(x - \hat{\mu}) \psi_{x+\hat{\mu}} + U_{\mu}^{\dagger}(x) \psi_{x-\hat{\mu}} \right] = E \psi_x. \quad (8)$$

We then solve Eq. (8) by taking a Fourier transformation. Note that the translation symmetry of this model is slightly modified from the usual lattice model due to the background field,

$$x_{2s-1} \sim x_{2s-1} + 1, \quad x_{2s} \sim x_{2s} + q_s. \quad (9)$$

Thus, writing the coordinate as $x_{2s} = q_s y_s + z_s$, the wavefunction is Fourier transformed as

$$\psi_x = \frac{1}{V} \sum_{k_1, \dots, k_{2r}} \exp \left[i \sum_{s=1}^r (k_{2s-1} x_{2s-1} + q_s k_{2s} y_s) \right] \tilde{\psi}_{z_1, \dots, z_r}(k_1, \dots, k_{2r}), \quad (10)$$

where V stands for the effective volume of the system, $V = L_1 \cdots L_{2r} / (q_1 \cdots q_r)$. The Schrödinger equation (8) is rewritten in this basis as

$$\sum_{s=1}^r \left[\tilde{\psi}_{z+\hat{s}} + \tilde{\psi}_{z-\hat{s}} + 2 \cos(k_{2s-1} - \omega_s z_s) \tilde{\psi}_z \right] = E \tilde{\psi}_z. \quad (11)$$

This is the higher-dimensional version of Harper's equation [23]. While a one-dimensional equation is obtained from the magnetic two-dimensional system, we have an r -dimensional equation from the $d = 2r$ theory. Furthermore, the matrix size of this higher-dimensional Harper's equation is $N_q \times N_q$ where $N_q \equiv q_1 \cdots q_r$. This means that the number of energy bands is just given by N_q , and there possibly exist $N_q - 1$ energy gaps. To discuss these energy gaps, one has to investigate a transfer matrix and its spectral curve associated with Harper's equation (11). In this case, however, it is impossible to show these energy bands are totally gapped with the same argument as the two-dimensional situation: we cannot apply a naive transfer matrix method, since there are still r directions in Eq. (11). Therefore it is difficult to obtain the corresponding *Hofstadter's butterfly* by diagonalizing Harper's equation (11): its *wings* are almost disappearing.

Let me comment on the relationship between the non-commutative space and Eq. (11). Indeed, the higher-dimensional tight-binding model, discussed in this section, can be represented in terms of the non-commutative torus. The coordinates of the two-dimensional non-commutative torus T_{θ}^2 are given by

$$UV = e^{2\pi i \theta} VU, \quad (12)$$

with $e^{2\pi i \theta}$ being the q th root of unity, $2\pi \theta \equiv \omega = 2\pi/q$. They can be written in $q \times q$ matrix forms,

$$U = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & & & & \\ & e^{i\omega} & & & \\ & & e^{2i\omega} & & \\ & & & \ddots & \\ & & & & e^{(q-1)i\omega} \end{pmatrix}. \quad (13)$$

Based on this description, we can consider the $d = 2r$ -dimensional non-commutative torus:

$$U_s V_t = e^{2\pi i \theta_{st}} V_s U_t \quad \text{for } s, t = 1, \dots, r. \quad (14)$$

In general, θ_{st} is a real symmetric matrix. In the case of (11), it is simply given by a diagonal matrix, $2\pi\theta_{st} = \omega_s\delta_{st}$. This means the non-commutativity on the $2r$ -dimensional torus is introduced to each two-dimensional subspace, $T_\theta^{2r} \rightarrow T_{\theta_1}^2 \times \cdots \times T_{\theta_r}^2$. Then the associated Hamiltonian can be written as

$$\mathcal{H}_{\text{tight}} = \sum_{s=1}^r \left[U_s + U_s^\dagger + V_s + V_s^\dagger \right]. \quad (15)$$

Although, precisely speaking, we have to include a factor corresponding to the plane wave, we now omit these factors for simplicity. This representation means that the non-commutative torus operator plays the role of the translation operator with the external magnetic field even in the higher-dimensional case.

2.1.1. π -flux state. Let us comment on the π -flux state in higher dimensions.¹ For the π -flux state [18,19], all the plaquettes have the same value, $P_{\mu\nu} = -1$, independent of its position x , and directions μ, ν as

$$P_{\mu\nu}(x) = U_\mu(x)U_\nu(x + \hat{\mu})U_\mu^\dagger(x + \hat{\nu})U_\nu^\dagger(x) = -1. \quad (16)$$

Such a configuration is realized by the following $\mathbb{Z}_2 \subset \text{U}(1)$ link variables,

$$U_\mu(x) = \eta_\mu \equiv (-1)^{x_1 + \cdots + x_{\mu-1}}. \quad (17)$$

Then the tight-binding Hamiltonian in the second quantized form with this gauge configuration yields

$$\mathcal{H}_{\text{tight}} = \sum_x \sum_{\mu=1}^d \eta_\mu c_x^\dagger (c_{x+\hat{\mu}} + c_{x-\hat{\mu}}). \quad (18)$$

We now consider the free field theory for simplicity. This is almost the same as the staggered Dirac operator [26–28]

$$\mathcal{S}_{\text{staggered}} = \sum_x \sum_{\mu=1}^d \frac{1}{2} \eta_\mu \bar{\chi}_x (\chi_{x+\hat{\mu}} - \chi_{x-\hat{\mu}}). \quad (19)$$

Actually, by applying the transformation $c_x \rightarrow i^{x_1 + \cdots + x_d} c_x$, the Hamiltonian (18) is rewritten as

$$\mathcal{H}_{\text{tight}} = \sum_x \sum_{\mu=1}^d i \eta_\mu c_x^\dagger (c_{x+\hat{\mu}} - c_{x-\hat{\mu}}). \quad (20)$$

This is just the staggered fermion action (19), up to a constant factor. Note that the staggered Dirac operator is anti-Hermitian while the tight-binding Hamiltonian is Hermitian.

There is no spinor structure in this formulation, but this staggered fermion is directly obtained from the naive Dirac fermion through the spin-diagonalization—see Appendix A for details. The sign factor $\eta_\mu = (-1)^{x_1 + \cdots + x_{\mu-1}}$ is a remnant of the gamma matrix. The staggered fermion enjoys an exact $\text{U}(1)$ chiral symmetry, which is generated by $\epsilon_x = (-1)^{x_1 + \cdots + x_d}$. This symmetry ensures its gapless excitation. Therefore the π -flux state is also generically gapless.

¹ The author is grateful to T. Misumi for pointing out an essential connection of this argument to that discussed in [24]. See also [25].

2.2. Dirac fermion models

We then attempt to extend Hofstadter's problem to the relativistic system. Similar approaches have been seen in the context of graphene [29–32].

The generic form of the Dirac fermion action is written as

$$\mathcal{S}_{\text{Dirac}} = \sum_{x,\mu} \left[\bar{\psi}_{x+\hat{\mu}} U_{\mu}(x) P_{\mu}^{+} \psi_x - \bar{\psi}_x U_{\mu}^{\dagger}(x) P_{\mu}^{-} \psi_{x+\hat{\mu}} \right] \equiv \sum_{x,y} \bar{\psi}_x D_{x,y} \psi_y. \quad (21)$$

Here P_{μ}^{\pm} stands for the spinor matrix, corresponding to a type of the lattice fermion.² The simplest one is given by

$$P_{\mu}^{\pm} = \frac{1}{2} \gamma_{\mu}. \quad (22)$$

This is just the naive lattice discretization of the relativistic fermion [34].³ We have other choices for the spinor matrix:

- Wilson fermion [34]:

$$P_{\mu}^{\pm} = \frac{1}{2} (\gamma_{\mu} \pm r \mathbf{1}). \quad (23)$$

- Karsten–Wilczek fermion [36,37]:

$$P_{\mu}^{\pm} = \frac{1}{2} \gamma_{\mu} \quad (\mu = d), \quad P_{\mu}^{\pm} = \frac{1}{2} (\gamma_{\mu} \pm ir \gamma_d) \quad (\text{otherwise}). \quad (24)$$

- Boriçi–Creutz fermion [38,39]:

$$P_{\mu}^{\pm} = \frac{1}{2} ((1 \pm ir) \gamma_{\mu} \mp ir \Gamma), \quad \Gamma = \frac{1}{2} \sum_{\mu=1}^d \gamma_{\mu}. \quad (25)$$

All these fermions include a free parameter r . We often consider the case of $r = 1$ for simplicity. Note that another lattice fermion, which is called the staggered fermion [26–28], is essentially equivalent to the naive fermion, and also the π -flux state as discussed in Sect. 2.1. The naive and staggered fermions are transformed to each other by the spin-diagonalization. Thus in this paper we do not deal with the staggered fermion explicitly.

The Dirac equation, which corresponds to the Schrödinger equation (8), is given by

$$\sum_{\mu=1}^d \left[U_{\mu}(x) P_{\mu}^{+} \psi_{x+\hat{\mu}} - U_{\mu}^{\dagger}(x - \hat{\mu}) P_{\mu}^{-} \psi_{x-\hat{\mu}} \right] = \lambda \psi_x. \quad (26)$$

When we write Eq. (26) as $D\psi = \lambda\psi$, although this Dirac operator D becomes non-Hermitian in general, one can define an alternative Hermitian operator, $H = \gamma_{d+1} D$. In the following we consider the spectrum of this Hermitian operator as $H\psi = E\psi$, instead of the original Dirac equation (26).

² For example, see [33].

³ As is well known, this simple lattice discretization scheme has a problem: there exist extra massless modes at low energy, which are called *species doublers*. To obtain a single chiral fermion we have to implement a much more complicated scheme, e.g. domain-wall fermion, overlap fermion, and so on (see, for example, the recent review in Ref. [35]). However, this problem does not concern our study because it does not affect the spectrum of the Dirac fermion.

We can also obtain the corresponding Harper's equation by taking a Fourier transformation. In this case, Harper's equation is slightly modified from (11) as

$$\sum_{s=1}^{d/2} \gamma_{d+1} \left[P_{2s}^+ \tilde{\psi}_{z+\hat{s}} - P_{2s}^- \tilde{\psi}_{z-\hat{s}} + \left(i \sin(k_{2s-1} - \omega_s z_s) (P_{2s-1}^+ + P_{2s-1}^-) + \cos(k_{2s-1} - \omega_s z_s) (P_{2s-1}^+ - P_{2s-1}^-) \right) \tilde{\psi}_z \right] = E \tilde{\psi}_z. \quad (27)$$

Here we have $2N_q$ energy bands due to the γ -matrix structure: they appear in a pair for the relativistic theory. In this case, it is again difficult to obtain a fully gapped spectrum since this reduced equation is not one-dimensional, but $d/2$ -dimensional.

3. Non-Abelian gauge field models

We consider another generalization of the Hofstadter problem in higher dimensions by applying a non-Abelian gauge field as a background configuration. In this paper we concentrate on the four-dimensional model with SU(2) gauge field. A further generalization to higher-dimensional and higher-rank systems seems to be straightforward.

3.1. Background configuration

We now introduce the following SU(2) background configuration as a simple generalization of the U(1) background (2),

$$A_0 = 0, \quad A_j = -\omega_j x_0 \sigma_j \quad \text{for } j = 1, 2, 3, \quad (28)$$

with σ_j being the Pauli matrix. Here we define filling fractions

$$\omega_j = \frac{2\pi}{L^2} n_j \equiv 2\pi \frac{p_j}{q_j}. \quad (29)$$

From this configuration we compute a field strength,

$$F_{0j} = -\omega_j \sigma_j, \quad F_{jk} = -\omega_j \omega_k x_0^2 \epsilon_{ijk} \sigma^i. \quad (30)$$

The total background flux is given by

$$\frac{1}{16\pi^2} \int_{L^4} d^4x \epsilon^{\mu\nu\rho\sigma} \text{Tr} F_{\mu\nu} F_{\rho\sigma} = \frac{6\pi^2}{L^6} N \int_{L^4} d^4x x_0^2 = 2\pi N. \quad (31)$$

Here this integral is taken over the four-dimensional hypercubic lattice of size L^4 , and we define the flux number $N = n_1 n_2 n_3$.

Thus the link variables associated with (28) are given as follows,

$$U_0(x) = \mathbb{1}, \quad U_j(x) = e^{-i\omega_j x_0 \sigma_j} = \mathbb{1} \cos \omega_j x_0 - i \sigma_j \sin \omega_j x_0. \quad (32)$$

In this case, the translation symmetry of the lattice system yields

$$x_0 \sim x_0 + q_{\text{LCM}}, \quad x_j \sim x_j + 1, \quad (33)$$

where q_{LCM} is the least common multiple of q_1, q_2 , and q_3 . This means that the unit cell of this system is also extended in only one dimension as well as the two-dimensional case with U(1) magnetic field.

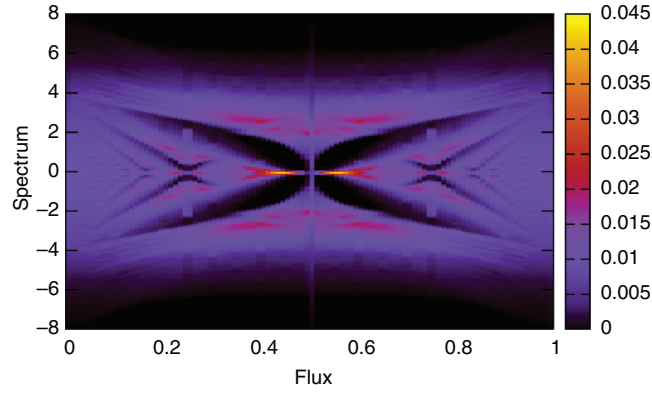


Fig. 1. Density of states for Harper's equation (34) of the flux $\omega_{j=1,2,3} = \omega_0 \equiv 2\pi p/q$ with $q = 100$, $p = 0, 1, 2, \dots, 99$.

3.2. Lattice fermion models

Let us first study the tight-binding model with the non-Abelian gauge field configuration. With a Fourier basis for $x_0 = q_{\text{LCM}}X_0 + \bar{x}_0$, the Hamiltonian (7) gives rise to the corresponding Harper's equation (11),

$$\tilde{\psi}_{\bar{x}_0+1} + \tilde{\psi}_{\bar{x}_0-1} + 2 \sum_{j=1}^3 (\cos k_j \cos \omega_j \bar{x}_0 + \sigma_j \sin k_j \sin \omega_j \bar{x}_0) \tilde{\psi}_{\bar{x}_0} = E \tilde{\psi}_{\bar{x}_0}. \quad (34)$$

This is just as effective a one-dimensional model as the standard two-dimensional Hofstadter problem. The original four-dimensional model is reduced to this due to $SU(2) \cong S^3$ gauge symmetry. On the other hand, in this case it is also difficult to show its spectrum is totally gapped as is the case for the $U(1)$ background in higher dimensions. Therefore we now study its density of states instead of the original energy spectrum.

Figure 1 shows the density of states for Harper's equation (34) against ω_0 , with homogeneous configuration $\omega_j = \omega_0$ for $j = 1, 2, 3$. Although its spectrum is not totally gapped, one can observe a hierarchical structure of the spectrum, which seems fractal at least based on this numerical computation. In order to numerically determine the corresponding fractal dimension, it is necessary to perform the calculation with a larger system size. At the even fraction flux, e.g. $\omega_0/(2\pi) = 1/2, 1/4, \dots$, we find a dip in the spectrum. A similar structure is discussed in the ordinary two-dimensional Hofstadter problem at the gapless point of the filling fraction.

In Fig. 2 we show the spectrum in the inhomogeneous flux with fixing ratios of $\omega_{j=1,2,3}$. When flux in some directions is turned off, e.g. $\vec{\omega} = \omega_0(1, 0, 0)$ and $\omega_0(1, 1, 0)$, the characteristic structure of Hofstadter's butterfly cannot be observed in the spectrum: the gap is completely closed. In the cases of $\vec{\omega} = \omega_0(2, 1, 1)$ and $\omega_0(2, 2, 1)$, there are some gaps in the spectrum and the hierarchical structure. We also find a dip at the gapless point, $\omega_0 = \pi$.

Let us now comment on the relation to the non-commutative space. We introduce the following non-commutative torus,

$$UV_j = e^{i\omega_j\sigma_j} V_j U \quad \text{for } j = 1, 2, 3. \quad (35)$$

Here U is defined as well as (13), while $V_{j=1,2,3}$ are slightly modified as

$$V_j = \text{diag} \left(1, e^{i\omega_j\sigma_j}, \dots, e^{(q_{\text{LCM}}-1)i\omega_j\sigma_j} \right). \quad (36)$$

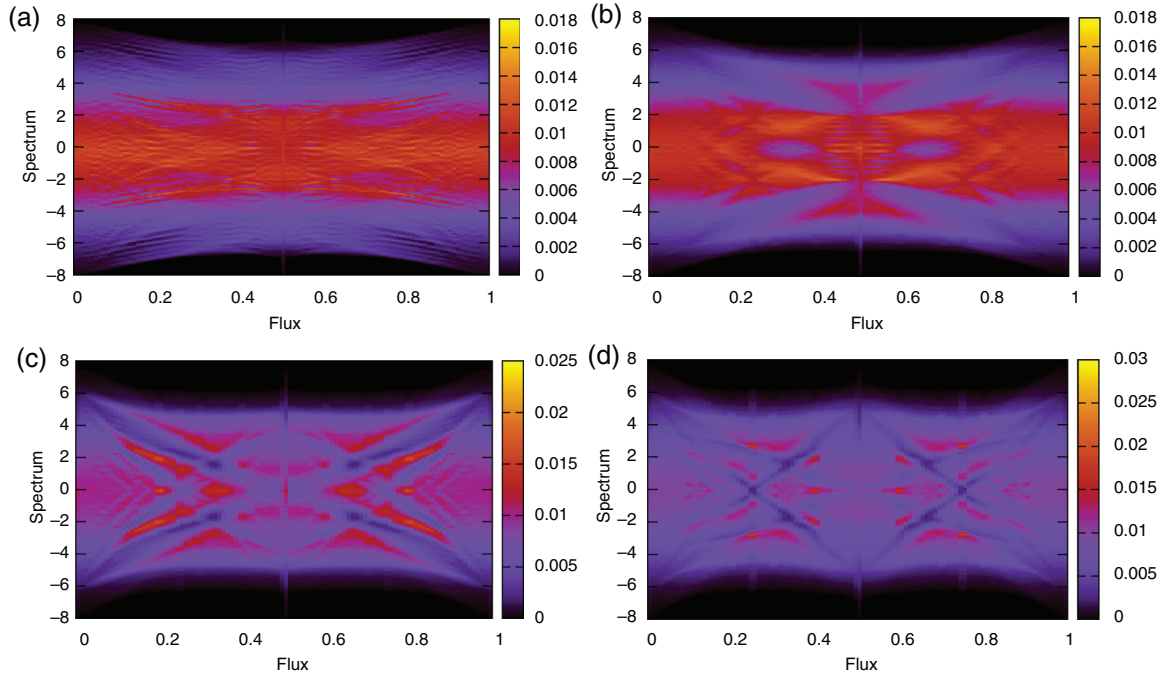


Fig. 2. Density of states for Harper's equation (34) with the inhomogeneous flux for $\vec{\omega} = \omega_0 \vec{n}$: (a) $\vec{n} = (1, 0, 0)$, (b) $(1, 1, 0)$, (c) $(2, 1, 1)$, and (d) $(2, 2, 1)$.

In this case the non-commutative parameter is matrix valued. Thus, the commutation relations between $V_{j=1,2,3}$ cannot be written in a simple way.

As discussed in Sect. 2, the tight-binding Hamiltonian with the external field, corresponding to Harper's equation (34), can be associated with the non-commutative torus. The naive form of the Hamiltonian without momentum dependence is given by

$$U + U^\dagger + \sum_{j=1}^3 [V_j + V_j^\dagger]. \quad (37)$$

However, since the second part is represented as

$$V_j + V_j^\dagger = \text{diag} (2, 2 \cos \omega_j, \dots, 2 \cos(q_{\text{LCM}} - 1)\omega_j) \times \mathbb{1}, \quad (38)$$

we cannot involve a matrix structure in this way. Actually the $\text{SU}(2)$ nature of the external field comes through the momentum dependence in (34).

We then investigate the Dirac fermion models as discussed in Sect. 2.2. Applying the background configuration (32), the corresponding Dirac equation is given by

$$\begin{aligned} \gamma_5 \left[P_0^+ \tilde{\psi}_{\vec{x}_0+1} - P_0^- \tilde{\psi}_{\vec{x}_0-1} + \sum_{j=1}^3 \left((\cos k_j \cos \omega_j \bar{x}_0 + \sigma_j \sin k_j \sin \omega_j \bar{x}_0) (P_j^+ - P_j^-) \right. \right. \\ \left. \left. + (\sin k_j \cos \omega_j \bar{x}_0 - \sigma_j \cos k_j \sin \omega_j \bar{x}_0) (P_j^+ + P_j^-) \right) \tilde{\psi}_{\vec{x}_0} \right] = E \tilde{\psi}_{\vec{x}_0}. \end{aligned} \quad (39)$$

Here we again show the Hermitian version of the Dirac equation by multiplying by the matrix γ_5 . When we write this Dirac operator in a matrix form, its matrix size is $8q_{\text{LCM}} = 2$ (color) \times 4 (spinor) \times q_{LCM} (flux). Here *color* corresponds to the rank of the gauge flux of $\text{SU}(2)$. On the other hand, the number of eigenvalues is given by $4q_{\text{LCM}}$, since each spectrum is doubly degenerated due to the spinor structure.

4. Summary and discussions

In this paper we have explored some extensions of the Hofstadter problem in higher dimensions. The first example is formulated with Abelian gauge configuration in $d = 2r$, giving rise to non-zero topological number. We have shown that half of the gauge potential can be trivial by applying the Landau gauge, thus the corresponding Harper's equation is essentially written as an r -dimensional lattice model.

We have also pointed out that the π -flux state is equivalent to the staggered formalism of the relativistic lattice fermion. The latter is directly related to the naive Dirac fermion through the spin diagonalization. This means the π -flux state involves the chiral symmetry, and thus it yields massless excitation in any dimensions.

We have then investigated SU(2) non-Abelian gauge field configuration in four dimensions. We have considered the configuration with one specific direction in four dimensions. In this case, hopping terms for the other three directions are reduced due to $SU(2) \cong S^3$ symmetry of the gauge field. Thus we have obtained the one-dimensional Harper's equation by utilizing the Fourier basis. We have calculated its spectrum numerically, and its hierarchical structure is actually observed.

Let us now comment on possibilities for future work along this direction. The two-dimensional Hofstadter problem is essentially related to the quantum group [29,40–42]: Harper's equation is directly regarded as Baxter's equation for the one-dimensional $U_q(\mathfrak{sl}_2)$ model. Thus it is interesting to explore the corresponding quantum group structure to the generalized Hofstadter problems discussed in this paper. In particular, the non-Abelian version of Harper's equation includes the matrix-valued coefficient. This corresponds to the q -parameter in the two-dimensional case, thus it is natural to investigate a kind of quantum group with matrix-valued q -parameter.

Next is the lattice study with various kinds of lattice fermions, i.e. Wilson [34], staggered [26–28], staggered Wilson [43–47], minimal-doubling [36–39,48–50], domain-wall [51–53] and overlap fermions [54]. They were originally introduced to tackle the difficulty of the chiral fermion on the lattice, but these formalisms themselves are interesting as statistical lattice models: some of them are actually investigated in the context of condensed-matter physics, for example, graphene, π -flux state, topological insulator/superconductor, and so on. Thus we hope the Hofstadter problem formulated with these lattice fermions is relevant to realistic condensed-matter physics.

It is also definitely interesting to consider the implications of the result obtained in this paper for realistic situations. An important difference between the non-Abelian gauge field and the U(1) magnetic field is whether it breaks the time-reversal symmetry of the system: the SU(2) gauge potential can be applied without breaking the time-reversal symmetry. It implies that, as a consequence of the dimensional reduction of the model discussed in this paper, one can possibly realize the non-Abelian Hofstadter system, for example, in topological insulators whose time-reversal symmetry is not broken. In addition, since there are already experimental techniques to realize the non-Abelian gauge potential and the ordinary Hofstadter system based on the U(1) field, respectively, one can expect that experimental realization of the non-Abelian Hofstadter system might be possible, especially in the cold atomic system.

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Appendix A. Spin diagonalization

We now show that there is an alternative expression for the d -dimensional naive Dirac fermion without spinor matrix structure. It is given by diagonalizing the corresponding γ matrices.

Let us start with the naive Dirac fermion on the lattice,

$$\mathcal{S} = \sum_{x,\mu} \left[\frac{1}{2} \bar{\psi}_x \gamma_\mu (\psi_{x+\hat{\mu}} - \psi_{x-\hat{\mu}}) + m \bar{\psi}_x \psi_x \right]. \quad (\text{A1})$$

Then, introducing the field χ_x defined as

$$\psi_x = \gamma_d^{x_d} \cdots \gamma_2^{x_2} \gamma_1^{x_1} \chi_x, \quad \bar{\psi}_x = \bar{\chi}_x \gamma_1^{x_1} \gamma_2^{x_2} \cdots \gamma_d^{x_d}, \quad (\text{A2})$$

we can represent the naive lattice fermion (A1) in the following form,

$$\mathcal{S} = \sum_{x,\mu} \left[\frac{1}{2} \eta_\mu(x) \bar{\chi}_x (\chi_{x+\hat{\mu}} - \chi_{x-\hat{\mu}}) + m \bar{\chi}_x \chi_x \right] \quad \text{with } \eta_\mu(x) = (-1)^{\sum_{\nu < \mu} x_\nu}. \quad (\text{A3})$$

Note that there is no spinor structure in this expression. In other words, the spinor matrix is diagonalized in this basis. This means the naive Dirac fermion can be rewritten in terms of a one-component fermionic field. This is just the staggered formalism of the relativistic fermion [27,28].

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